What is the connection between ballistic deposition and the Kardar-Parisi-Zhang equation?

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Ballistic deposition (BD) is believed to belong to the Kardar-Parisi-Zhang (KPZ) universality class. In this paper we study the validity of this belief by rigorously deriving a continuum equation from the BD microscopic rules, which deviates from the KPZ equation. We show that in one dimension and in the presence of noise the deviation is not important. This is not the case in the absence of noise. In more than one dimension and in the presence of noise we obtain an equation that superficially seems to be a continuum equation but in which the symmetry under rotations around the growth direction is broken.

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I. INTRODUCTION

Kinetic roughening of nonequilibrium surface growth has been of great interest in recent years. The kinetic growth processes have been intensively studied via various discrete models and continuum equations and exhibit nontrivial scaling behavior [1–5]. The surface width W, which is the standard deviation of the surface height, scales as $W(L,t) \sim L^{\alpha} g(t/L^{z})$, where the scaling function $g(u)$ is constant for $u \gg 1$ and behaves like u^{β} for $u \ll 1$. Since the growth is independent of the system size L at the beginning of the process, the exponents must obey the scaling relation $\beta = \alpha/z$. The scaling behavior of the growth is characterized by the roughness exponent α, the growth exponent β, and the dynamical exponent z, and these exponents determine the universality class.

The discrete ballistic deposition (BD) growth model [2,6,7] and the Kardar-Parisi-Zhang (KPZ) equation [8] describing continuous growth, to be described in the following, are believed to belong to the same universality class. Such a relation can be exploited in many ways. For example, simulations of the BD system that are comparatively fast may serve to describe the numerically more demanding KPZ system. What is the evidence that the two models belong to the same universality class? A number of ways may be considered to show that two models belong to the same universality class. The most direct is to show that the two models correspond to the same fixed point system. (In the case under consideration it is not a Hamiltonian that describes the system.) To the best of our knowledge, this has never been done for BD and KPZ models. Even for systems where the study is much more established, showing that two models are in the same universality class is not entirely trivial. Another possible way is to assert that two models of the same dimension possessing the same symmetries and range of interaction are in the same universality class [18,19]. This is based also on scaling arguments that check the relevance of various possible terms in the hydrodynamic limit [2]. It is clear that in one dimension the BD and KPZ models have the same symmetries and short-range "interactions" and therefore should belong to the same universality class. In higher dimensions the lattice structure on which the BD model is defined may break rotation symmetry around the growth direction. At first sight this may not seem to be a problem. An Ising model is also defined on a lattice but in the long-wavelength limit the rotational symmetry is recovered and the model is equivalent to a field theory which is rotationally invariant. We will come back to this point later. The third way is to use massive simulations to obtain and compare the exponents of the two systems. (This is somewhat self-defeating as the whole usefulness of the concept of universality class is in predicting the exponents in one model given the exponents of another. Still in some cases it may be the only practical hope to determine as a matter of principle if two models belong to the same universality class.) Table I presents the exponents obtained by the BD model in one, two, and three dimensions by various authors.

In one dimension the exact values of the KPZ exponents are known to be $\alpha = 1/2$ and $\beta = 1/3$. The results for BD are scattered but two of the results [11,12] are close enough to the exact KPZ values. In two dimensions the values of the KPZ systems are not known exactly. There is a large diver-

TABLE I. Scaling exponents obtained by simulations for the ballistic deposition model for $d=1$, 2, and 3 dimensions.

d	α	β	z	Reference
1+1	0.42	0.3		[9]
1+1		0.3		[10]
1+1	0.47	0.33		[11]
1+1	0.506	0.339		[12]
1+1	0.45	0.32	1.40	[13]
2+1	0.33	0.24		[10]
2+1	0.3	0.22		[11]
2+1	0.35	0.21		[14]
2+1	0.36			[12]
2+1	0.26	0.21	1.24	[13]
2+1	0.38	0.229	1.62	[15]
3+1	0.12			[13]

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sity of results that range between 0.18 and 0.4 for α and 0.1 and 0.25 for β [16] (these results were obtained by a direct integration of the KPZ equation or the equivalent directed polymer problem). However, it usually accepted that $\alpha = \sim 0.4$ and $\beta = 0.24$ are the KPZ exponents in 2+1 dimensions. This result is also very close to that of Forrest and Tang who simulated the single-step solid-on-solid (SOS) model (which is believed to be in the KPZ universality class) [17] and got $\alpha = 0.385$ and $\beta = 0.240$.

Again the results for the BD model are very scattered, and although there is one result [10] that is close enough for β to the above-mentioned results, the scaling relation $\beta = \alpha/(2-\alpha)$ is only poorly obtained. This is consistently true also for all the other results in two dimensions. Can the results of BD simulations serve as serious evidence that the BD model is in the universality class of the KPZ model? In one dimension our answer is maybe. In more than one dimension our answer is no. This is not to say whether indeed the BD model is in a different universality class or that the results of simulations are not good enough or both. (In fact the large scatter of the results suggests that at least some of the simulations are indeed not good enough.) This should be compared with the very accurate exponents obtained in the study of critical phenomena of translational invariant systems. For those systems results of very different methods, such as high-temperature series, momentum-space renormalization, and real-space renormalization, are known to converge on the same values with much higher accuracy.

This unfortunate situation suggests a fourth route in which the question might be answered and this is a direct derivation of the KPZ equation from the BD model in the continuum limit. Before discussing previous work in this direction and our present one, we first describe in short the two models under consideration.

The BD model in 1+1 dimensions (actually on a two-dimensional square lattice) can be described as follows. At time t, the height of the interface at site i is $h_i(t)$. We choose a random position above the surface and allow a particle to fall vertically toward it. The particle sticks to the first site along the trajectory that has an occupied nearest neighbor. If no such neighbor exists, it lands on the surface below. Actually, the version just described is called the nearest-neighbor (NN) BD model. In another version of the model, also known as the next-nearest-neighbor (NNN) BD model [2], which will interest us in this paper, the particle is allowed to stick to a diagonal neighbor as well, as shown in Fig. 1. At time $t+1$, a column i is chosen at random, and the height $h_i(t+1)$ is then given by

$$h_i(t+1) = \max\{h_{i-1}(t), h_i(t), h_{i+1}(t)\} + 1. \tag{1}$$

As mentioned above, the continuum equation that is believed to capture the essential dynamics of the BD models is the famous KPZ [8] equation, given by

$$\frac{\partial h}{\partial t}(\vec{r},t) = \nu \nabla^2 h(\vec{r},t) + \frac{\lambda}{2}(\nabla h)^2 + \eta(\vec{r},t), \tag{2}$$

where $h(\vec{r},t)$ is height of the interface at the point $\vec{r}$ and time t, and $\eta(\vec{r},t)$ is a noise term such that

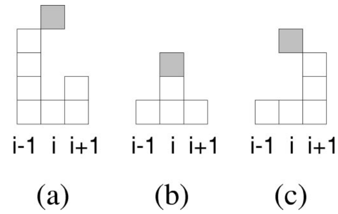

FIG. 1. Schematic representation of the next-nearest-neighbor ballistic deposition (NNN BD) model. A particle falls vertically and sticks to the first site along its trajectory that has an occupied nearest neighbor. The particle is allowed to stick to a diagonal neighbor as well.

$$\langle \eta(\vec{r},t) \rangle = 0,$$

$$\langle \eta(\vec{r},t)\eta(\vec{r}\,',t') \rangle = 2D_0 \delta(\vec{r}-\vec{r}\,')\delta(t-t'). \tag{3}$$

In the last few years most of the efforts in establishing direct connections between discrete models and continuum equations have been directed towards developing a general procedure via formal expansions of discrete equations of motion [20,22–31]. Usually, the derivation of the continuum equation is based on regularizing and coarse graining discrete Langevin equations that are obtained from a Kramers-Moyal expansion of the master equation. In simple words, transition probabilities are calculated from the microscopic rules of the model for any given discrete height configuration $\{h_i\}$. These expressions usually contain discrete θ (Heaviside) and δ (Dirac) functions. But since the transition probabilities are supposed to be continuous functions (so that the expansion that is used there is meaningful), some coarse-graining procedure is needed. More specifically, this involves expansions of the form

$$\theta(x) = 1 + \sum_{k=1}^{\infty} A_k x^k, \tag{4}$$

as originally suggested in [23]. Sometimes a less restricted form $\theta(x) = 1 + \sum_{k=0}^{\infty} A_k x^k$ is used. Other suggestions include

$$\theta(x) = [1 + \tanh(Cx)]/2, \tag{5}$$

where C is an arbitrary positive parameter, with the exact $\theta(x)$ function being obtained in the limit $C \to \infty$ [26]. C is then used in the expansion as an uncontrolled parameter. Others [24] use the shifted form $\theta(x) = \lim_{C\to\infty}[1+\tanh(C\{x+\alpha\})]/2$ with $\alpha \in (0, \frac{1}{2}]$ or a modified version using $\arctan(Cx)$ [27] or $\mathrm{erf}(Cx)$ [29] instead of $\tanh(Cx)$ (each version has its advantages) in Eq. (5).

In some cases the master equation approach is problematic, as in the case of deriving the KPZ equation [8] from the ballistic deposition model [25]. Thus a similar approach, yet more appropriate for that specific case was developed. The method is based on dealing directly with the discrete Langevin equation rather than with its associated master equation. Still, expansions like

$$\theta(x-a) = \theta(x) + \sum_{n=1}^{\infty} \frac{a^n}{n!} \left. \frac{\partial^n \theta(y)}{\partial y^n} \right|_{y=x} \tag{6}$$

were used. A different, yet closely related Langevin-based approach used the following representation of the max function [29,30]:

$$\max\{A,B,C\} = \lim_{\varepsilon \to 0^+} \varepsilon \ln\{e^{A/\varepsilon} + e^{B/\varepsilon} + e^{C/\varepsilon}\}, \tag{7}$$

in order to go from the discrete BD model to the continuum KPZ equation. Lately [31], influenced by the last representation, the Edwards-Wilkinson [1] equation was derived from a discrete model using

$$\theta(x) = \max\{x+a,0\} - \max\{x\} = \lim_{\varepsilon \to 0^+} \left\{ \frac{\varepsilon}{a} \ln\left[\frac{e^{(x+a)/\varepsilon}+1}{e^{x/\varepsilon}+1} \right] \right\}, \tag{8}$$

where a is any constant in the interval (0, 1].

In spite of these many new and interesting derivations, one can easily point out three main drawbacks of this last approach. First, in many cases the derivation is performed in one dimension, where higher dimensions are not discussed at all or, specifically, known to cause fatal difficulties (see Ref. [30], for example).

Second, obviously the mathematics used in the derivation is not so rigorous. For example, an expansion like the one given in Eq. (4) is problematic because the Heaviside function is certainly not analytic around zero. Another example is when taking an expression like Eq. (5) and expanding it for small C—while the limiting procedure that is needed for the equality to hold requires $C \to \infty$.

Third, since artificial parameters like C and ϵ that cannot always be removed later are present it is not possible to infer the macroscopic quantities (such as the diffusion coefficient) from the microscopic rules.

In what follows we will question the relation between the KPZ and BD models and claim that it is not a coincidence that such a formal (and "rigorous") derivation was not found. In Sec. II we will show that strictly speaking the continuum model that describes BD in one dimension is not the KPZ equation, but rather an equation with $|\nabla h|$ instead of the quadratic $(\nabla h)^2$ term. Then, we will claim that in the case of stochastically driven growth, the difference between the two equations is not dramatic and just modifies the value of the coupling constant in the KPZ equation. In Sec. III we show (using [32,33]) that when the noise is shut down and the dynamics becomes deterministic, then the difference between the BD and KPZ models is very important. In Sec. IV, we make a step forward towards a derivation of a continuum equation that emanates from the BD model in higher dimensions. However, the equation we find reflects the underlying structure of the discrete lattice, in the sense that the equation depends on the directions of the coordinate system induced by the lattice. Therefore, the equation we obtain, which breaks rotational symmetry, cannot be conclusively related to the KPZ equation. At the end, in Sec. V, a brief summary of the results obtained in this paper is presented.

II. DERIVATION OF A CONTINUUM EQUATION IN ONE DIMENSION

We begin with the one-dimensional discrete model

$$h_i(t+1) = \max\{h_{i-1}(t), h_i(t), h_{i+1}(t)\} + \eta_i(t), \tag{9}$$

where $h_i(t)$ is the height of a surface at lattice position i and time t (the time is discrete as well). In addition, $\eta_i(t)$ is a white noise term satisfying

$$\langle \eta_i(t) \rangle = 0,$$

$$\langle \eta_i(t)\eta_{i'}(t') \rangle = 2D_0 \delta_{i,i'} \delta_{t,t'}. \tag{10}$$

This discrete model is a formal expression for an n-particle NNN BD model [see Eq. (21) in Ref. [30]]. It is well known that the fact that here we deposit more than one particle per unit time is not important for the universal behavior of the model. Actually the same discrete model is used to describe formation of foam [6].

First we represent the max{} operator using Heaviside functions

$$\max\{a,b\} = a\theta(a-b) + b\theta(b-a). \tag{11}$$

Thus, for three arguments we also get

$$\max\{a,b,c\} = a\theta(a-b)\theta(a-c) + b\theta(b-a)\theta(b-c) + c\theta(c-a)\theta(c-b). \tag{12}$$

Now we represent the Heaviside function using the sign function

$$\theta(x) = \frac{1}{2} + \frac{1}{2}\mathrm{sgn}(x). \tag{13}$$

Applying these representations to Eq. (9) we get

$$\max\{h_{i-1}(t),h_i(t),h_{i+1}(t)\} = h_{i-1}(t)\frac{1}{4}\{1+\mathrm{sgn}[h_{i-1}(t)-h_i(t)]\}\{1+\mathrm{sgn}[h_{i-1}(t)-h_{i+1}(t)]\} + h_i(t)\frac{1}{4}\{1+\mathrm{sgn}[h_i(t)-h_{i-1}(t)]\}\{1 + \mathrm{sgn}[h_i(t)-h_{i+1}(t)]\} + h_{i+1}(t)\frac{1}{4}\{1+\mathrm{sgn}[h_{i+1}(t)-h_{i-1}(t)]\}\{1+\mathrm{sgn}[h_{i+1}(t)-h_i(t)]\}. \tag{14}$$

Thus, after some simple algebra,

$$h_i(t+1)-h_i(t)=\frac{1}{4}[h_{i-1}(t)-2h_i(t)+h_{i+1}(t)]-\frac{1}{4}h_i(t)+\frac{1}{4}[h_i(t)-h_{i-1}(t)]\mathrm{sgn}[h_i(t)-h_{i-1}(t)]+\frac{1}{4}[h_{i+1}(t)-h_{i-1}(t)]\mathrm{sgn}[h_{i+1}(t)-h_{i-1}(t)]+\frac{1}{4}[h_{i+1}(t)-h_i(t)]\mathrm{sgn}[h_{i+1}(t)-h_i(t)]+\frac{1}{4}h_{i-1}(t)\mathrm{sgn}[h_i(t)-h_{i-1}(t)]\mathrm{sgn}[h_{i+1}(t)-h_{i-1}(t)]-\frac{1}{4}h_i(t)\mathrm{sgn}[h_i(t)-h_{i-1}(t)]\mathrm{sgn}[h_{i+1}(t)-h_i(t)]+\frac{1}{4}h_{i+1}(t)\mathrm{sgn}[h_{i+1}(t)-h_{i-1}(t)]\mathrm{sgn}[h_{i+1}(t)-h_i(t)]+\eta_i(t). \quad (15)$$

The last expression is written in such a way that identifying discrete derivatives is easy. Therefore, denoting the spatial and temporal increments by Δx and Δt, and using the simplifying fact $\mathrm{sgn}(ax)=\mathrm{sgn}(x)$ (when $a>0$), we get

$$\frac{\partial h}{\partial t}(x,t)=\frac{(\Delta x)^2}{4\Delta t}\left[1+\mathrm{sgn}^2\left(\frac{\partial h}{\partial x}\right)\right]\frac{\partial^2 h}{\partial x^2}(x,t)+\frac{\Delta x}{\Delta t}\left|\frac{\partial h}{\partial x}\right|+\frac{1}{\Delta t}\eta(x,t). \quad (16)$$

Note that

$$\mathrm{sgn}^2(x)=\begin{cases}1, & x\neq 0,\\ 0, & x=0.\end{cases}$$

Thus, for a strictly nonsmooth surface (i.e., a surface that is not flat), almost everywhere we can use the replacement $\mathrm{sgn}^2(\partial h/\partial x)=1$ in order to further simplify Eq. (16). In addition, since in the discrete model we actually took $\Delta x=\Delta t=1$, we get the final result

$$\frac{\partial h}{\partial t}(x,t)=\frac{1}{2}\frac{\partial^2 h}{\partial x^2}(x,t)+\left|\frac{\partial h}{\partial x}\right|+\eta(x,t). \quad (17)$$

This equation is a result of a straightforward exact coarse graining of Eq. (9). Thus, one can be convinced that the BD model (at least this n-particle version of it) does not lead exactly to the KPZ equation. This fact might explain the difficulties encountered in the past while deriving KPZ equation from the BD model.

Symmetry arguments suggest that our one-dimensional system (17) is indeed in the universality class of the KPZ equation. This is supported by the results of Ref. [33], which studied Langevin equations (of the KPZ type) with a generalized $|\nabla h|^\mu$ nonlinearity. They concluded that no matter what value of μ was taken, KPZ behavior was revealed, as long as the noise term is present (the case of no noise is interesting by itself, and will be discussed in Sec. III below). Thus, this fundamental observation of Amar and Family [33] establishes the link between Eq. (17) and the KPZ equation, by simply identifying $\mu=1$ in that equation. Hence, the route between the BD and KPZ models in one dimension is now understood, and the common knowledge about the BD model is justified.

Now, it would be interesting to complete the picture and to extract the macroscopic coefficients that describe the continuum KPZ equation that is related to the BD model. First, the diffusion coefficient can be inferred easily from Eq. (17). Then, the noise amplitude is unaffected by the coarse graining. Finally, we would like to identify the coupling constant λ in the KPZ equation (2). However, since the nonlinear term in Eq. (17) is not in a KPZ form, λ cannot be just read from the equation. Thus, we push further the idea presented in the previous paragraph and make the identification

$$\frac{|\nabla h|}{\langle|\nabla h|\rangle}\simeq\frac{|\nabla h|^2}{\langle|\nabla h|^2\rangle}, \quad (18)$$

where $\langle\cdots\rangle$ means steady-state averaging. The last equation leads to $|\nabla h|\simeq(\langle|\nabla h|\rangle/\langle|\nabla h|^2\rangle)(\nabla h)^2$. Plugging this estimate into Eq. (17) gives the prediction $\lambda/2\simeq\langle|\nabla h|\rangle/\langle|\nabla h|^2\rangle$ for the coupling constant of the KPZ equation (2). The meaning of this expression is that the KPZ coupling constant that describes the surface that grows under the BD model is obtained by calculating the two quantities $\langle|\nabla h|\rangle$ and $\langle|\nabla h|^2\rangle$ in a BD simulation (in steady state) and plugging the obtained averages into the expression given above for $\lambda/2$.

The last prediction has been checked numerically by us. We grew a one-dimensional interface on a discrete lattice with linear size $L=1024$ using the microscopic rules of the BD model [given by Eq. (9) above]. Using this simulation we found $\langle|\nabla h|\rangle=0.4705$ and $\langle|\nabla h|^2\rangle=0.4639$, which leads to the prediction $\lambda\simeq 2.17$. Using an inverse method—that is, "guessing" the best macroscopic parameters that recover the same growth process (see Refs. [2,21])—we found $\lambda=2.27$, which supports the rough estimate of λ we suggested above.

Note that if we wanted to use the same kind of replacement of $|\nabla h|$ by $(\nabla h)^2$ not in steady state but say at the beginning of the growth, the coefficient would have to be a time-dependent function that will eventually approach the steady-state value that follows from Eq. (18). Thus it might be tempting to speculate that the short-time dependence of the coefficient might be the reason for some problems encountered in simulations in the numerical determination of the exponent β.

Up to now we have focused on the common characteristics of the BD and KPZ models. However, as will be seen in the following two sections, if the noise is turned off or if we go to higher dimensions, significant differences between the two models can be found.

III. DETERMINISTIC FLATTENING

In this section we discuss the dynamics of the discrete BD model and the continuum model with $|\nabla h|^\mu$ nonlinearity for

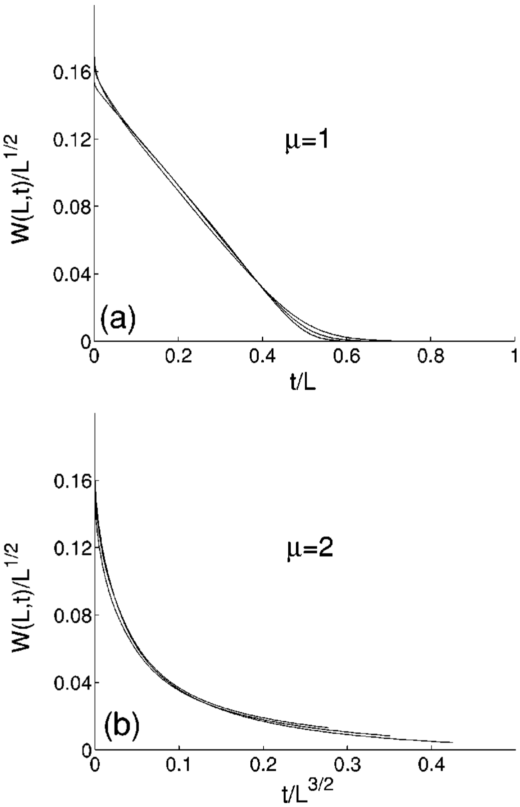

FIG. 2. Scaling plot for the deterministic equation (19) for $L=256$, 512, and 1024 with (a) $\mu=1$ using $z_d=1$ and with (b) $\mu=2$ using $z_d=\frac{3}{2}$.

$\mu=2$ (KPZ) and for $\mu=1$, when the noise term that appears in them is eliminated. This limit mimics the physical scenario when the deposited materials stop falling, and the interface relaxes to a flat surface, due to its diffusive term. In the previous section we showed that in one dimension the KPZ model and the continuum version of the BD model are equivalent in steady state, where the flattening of the surface due to diffusion is balanced by the external noise. It would be also interesting to study the relation between the two models in the absence of noise.

The starting point of this discussion is the observation of Krug and Spohn [32] regarding a deterministic continuum equation of the general form

$$\frac{\partial h}{\partial t} = \nu\nabla^2 h + g|\nabla h|^{\mu} \tag{19}$$

(where $\mu \geqslant 1$), which describes the smoothing of an initially rough surface under deterministic growth. They argued that if the initial surface has roughness exponent α, the scaling relation

$$z_d = \min\{2, \mu(1-\alpha)+\alpha\} \tag{20}$$

should hold, where z_d is a dynamic exponent for deterministic evolution, which, similar to the stochastically driven case, determines the time scale of the decaying width (see Fig. 2).

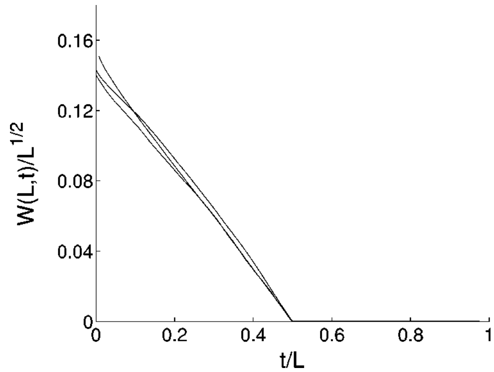

FIG. 3. Scaling plot for the deterministic NNN BD model given by Eq. (9) for $L=256$, 512, and 1024 using $z_d=1$.

Influenced by this work, Amar and Family [33] made a systematic study of such surface growth equations with a generalized nonlinearity and found good agreement between the scaling relation and the numerical integration of Eq. (19). We also performed a numerical integration, using the same integration scheme as in Ref. [33] [the coefficients ν and g were taken from Eq. (17) and using the result of the previous section] for the cases $\mu=1$ [Eq. (17)] and $\mu=2$ (KPZ) in 1+1 dimensions. We also used a rough surface with $\alpha=\frac{1}{2}$ as an initial condition. A scaling plot of our results for the surface width $W(L,t)$ for $\mu=1$ versus t/L_d^z with $z_d=1$ is given in Fig. 2(a), and a similar scaling plot for $\mu=2$ with $z_d=\frac{3}{2}$ is shown in Fig. 2(b).

As one can see, the scaling is good in both cases, in agreement with the scaling relation (20) and therefore consistent with Refs. [32,33]. Thus, the difference between the deterministic dynamics of the KPZ equation and that of Eq. (17) is now self-evident and not new. Two questions should be asked here. Our derivation of Eq. (17) does not depend on the presence of noise. Therefore, we might expect that even when deterministic dynamics is considered, discrete BD will be equivalent to the $\mu=1$ case rather than to the $\mu=2$ case (KPZ). The first question is whether this is indeed the case. The second question regards the predicted values for the diffusion coefficient (namely, $\nu=\frac{1}{2}$) and the coupling (i.e., the coefficient of the $|\nabla h|^{\mu}$ term—namely, $g=1$) for the n-particle NNN BD model that we read from Eq. (17). The question is whether this prediction is a sensible one.

In order to answer these questions we implemented the NNN BD model [whose sticking rules are given by Eq. (9)] on a two-dimensional square lattice (1+1 dimensions) and measured the surface width $W(L,t)$ of the flattening surface, using a rough surface with $\alpha=\frac{1}{2}$ as an initial condition (just like the continuum models above). We present the results in Fig. 3.

As can be seen in Fig. 3, the deterministic dynamics of the discrete model is evidently the one we have seen for the $\mu=1$ continuum equation, since the data collapse in the scaling plot with $z_d=1$ is good. Moreover, Figs. 2(a) and 3 look pretty much the same. The main difference between the two is the rounded tail in Fig. 2(a) around $t/L=0.5$ compared to the sharp transition in Fig. 3. This difference can be accounted for by the fact that the numerical integration of the continuum equation used a small time increment ($\Delta t=0.05$) while the discrete growth model used a much larger one ($\Delta t=1$). This implies a better temporal resolution around $t/L=0.5$ by numerical integration when compared to the simulation. In addition, the similarity between the two scaling plots means that the predicted macroscopic quantities are consistent with the results obtained from the simulation.

In this section we studied the difference between the deterministic versions of the BD and KPZ models. We also showed that Eq. (17) captures the dynamical behavior of the BD model even in the deterministic regime. This discussion served another purpose—namely, of providing a numerical support for the ν and λ we found from the formal derivation.

IV. RESULTS FOR HIGHER DIMENSIONS

In this section we generalize the formal derivation given in Sec. II to higher dimensions. As will be seen, the derivation is not exactly the same, and the resulting continuum equation is not a simple generalization of the one-dimensional equation (17).

We begin with the n-particle NNN BD model in d dimensions (discretized on a cubic hyperlattice):

$$h_{\vec{r}}(t+1) = \max\{h_{\vec{r}}(t), h_{\vec{r}+\hat{x}_1}(t), h_{\vec{r}-\hat{x}_1}(t), \ldots, h_{\vec{r}+\hat{x}_d}(t), h_{\vec{r}-\hat{x}_d}(t)\} + \eta_{\vec{r}}(t), \qquad (21)$$

where $\hat{x}_i$ is a unit vector in the ith direction.

As in Sec. II, using Eqs. (12) and (13), coarse graining (space), and using the simplifications $\operatorname{sgn}(ax)=\operatorname{sgn}(x)$ (for $a\neq 0$) and $\operatorname{sgn}^2(\partial h/\partial x_i)=1$ (which is again true almost anywhere), we get

$$2^{2d}h_{\vec{r}}(t+1) = 2\sum_{i=1}^{d}\left\{ h_{\vec{r}+\hat{x}_i}(t)\left[1+\operatorname{sgn}\left(\frac{\partial h}{\partial x_i}\right)\right]\prod_{\substack{j=1\\ j\neq i}}^{d}\left[1+\operatorname{sgn}\left(\frac{\partial h}{\partial x_i}-\frac{\partial h}{\partial x_j}\right)\right]\left[1+\operatorname{sgn}\left(\frac{\partial h}{\partial x_i}+\frac{\partial h}{\partial x_j}\right)\right] + h_{\vec{r}-\hat{x}_i}(t)\left[1-\operatorname{sgn}\left(\frac{\partial h}{\partial x_i}\right)\right] \times\prod_{\substack{j=1\\ j\neq i}}^{d}\left[1-\operatorname{sgn}\left(\frac{\partial h}{\partial x_i}-\frac{\partial h}{\partial x_j}\right)\right]\left[1-\operatorname{sgn}\left(\frac{\partial h}{\partial x_i}+\frac{\partial h}{\partial x_j}\right)\right]\right\} + \eta_{\vec{r}}(t). \qquad (22)$$

Using $1\pm\operatorname{sgn}(x)=2\theta(\pm x)$ we get

$$h_{\vec{r}}(t+1) = \sum_{i=1}^{d}\left\{ h_{\vec{r}+\hat{x}_i}(t)\,\theta\left(\frac{\partial h}{\partial x_i}\right) \times\prod_{\substack{j=1\\ j\neq i}}^{d}\left[\theta\left(\frac{\partial h}{\partial x_i}-\frac{\partial h}{\partial x_j}\right)\theta\left(\frac{\partial h}{\partial x_i}+\frac{\partial h}{\partial x_j}\right)\right] + h_{\vec{r}-\hat{x}_i}(t)\,\theta\left(-\frac{\partial h}{\partial x_i}\right)\prod_{\substack{j=1\\ j\neq i}}^{d}\left[\theta\left(-\frac{\partial h}{\partial x_i}+\frac{\partial h}{\partial x_j}\right) \times\theta\left(-\frac{\partial h}{\partial x_i}-\frac{\partial h}{\partial x_j}\right)\right]\right\} + \hat{\eta}_{\vec{r}}(t), \qquad (23)$$

where $\hat{\eta}_{\vec{r}}(t)=2^{-2d}\eta_{\vec{r}}(t)$. Notice that

$$\theta(a)\,\theta(a+x)\,\theta(a-x) = \theta(a)\,\theta(a-|x|) = \theta(a)\,\theta(|a|-|x|), \qquad (24)$$

and thus

$$h_{\vec{r}}(t+1) = \sum_{i=1}^{d}\prod_{\substack{j=1\\ j\neq i}}^{d}\theta\left(\left|\frac{\partial h}{\partial x_i}\right|-\left|\frac{\partial h}{\partial x_j}\right|\right)\left[h_{\vec{r}+\hat{x}_i}(t)\,\theta\left(\frac{\partial h}{\partial x_i}\right) + h_{\vec{r}-\hat{x}_i}(t)\,\theta\left(-\frac{\partial h}{\partial x_i}\right)\right] + \hat{\eta}_{\vec{r}}(t). \qquad (25)$$

Using once again the relation $\theta(x)=\frac{1}{2}[1+\operatorname{sgn}(x)]$ and reorganizing the last equation leads us to

$$h_{\vec{r}}(t+1) = \frac{1}{2}\sum_{i=1}^{d}\prod_{\substack{j=1\\ j\neq i}}^{d}\theta\left(\left|\frac{\partial h}{\partial x_i}\right|-\left|\frac{\partial h}{\partial x_j}\right|\right)\left[h_{\vec{r}+\hat{x}_i}(t)-2h_{\vec{r}}(t) + h_{\vec{r}-\hat{x}_i}(t)+2\frac{\partial h}{\partial x_i}\operatorname{sgn}\left(\frac{\partial h}{\partial x_i}\right)+2h_{\vec{r}}(t)\right] + \hat{\eta}_{\vec{r}}(t), \qquad (26)$$

Coarse graining in both space and time gives us the final result

$$\frac{\partial h}{\partial t}(\vec{r},t) = \sum_{i=1}^{d}\left(\frac{1}{2}\frac{\partial^2 h}{\partial x_i^2}+\left|\frac{\partial h}{\partial x_i}\right|\right)\prod_{\substack{j=1\\ j\neq i}}^{d}\theta\left(\left|\frac{\partial h}{\partial x_i}\right|-\left|\frac{\partial h}{\partial x_j}\right|\right) + \hat{\eta}(\vec{r},t). \qquad (27)$$

This equation might seem messy for a second. However, it has a very simple structure. First, for $d=1$ it collapses trivially to Eq. (17). Then, in higher dimensions, the expression

$$\prod_{\substack{j=1 \\ j\neq i}}^{d} \theta\left(\left| \frac{\partial h}{\partial x_i} \right| - \left| \frac{\partial h}{\partial x_j} \right| \right)$$

picks the direction m in space along which the gradient is maximal, and the considered part of the local growth rate has a contribution from that direction alone,

$$\frac{1}{2}\frac{\partial^2 h}{\partial x_m^2} + \left| \frac{\partial h}{\partial x_m} \right| .$$

Actually, taking a glance at the original discrete model, this result is not such a surprise since both formulations contain this "maximal" growth ingredient in them.

The above is in a sense not really a continuum equation in dimension larger than 1, since the directions of original lattice axes are still present and the symmetry under rotations present in the KPZ model is not present in our Eq. (27). This means that the two models have different symmetries and therefore there is no *a priori* reason for them to belong to the same universality class. (This situation is very different from what happens, say, in the Ising model, where the discrete $\Sigma_{n(i)}[\sigma_{n(i)}-\sigma_i]^2$ [$n(i)$ being a nearest neighbor of i], which is not rotationally invariant, is safely replaced by $[\nabla\sigma]^2$, which is rotationally invariant.) The situation as we see it is the following: None of the four suggested routes to check whether the two models belong to the same universality class yields a convincing answer. To our knowledge a fixed point system corresponding to both systems is not known. In spite of the fact that a BD simulation is relatively simple, it seems that massive work was done only in one and two dimensions. In two dimensions the scatter of the results and the violation of the scaling relation connecting α and β cannot really be interpreted as the exponents being those of the KPZ system. The symmetry argument does not work as there is a symmetry not shared by the two models. Our final conclusion is that we do not see, at present, any compelling reason for the two models to belong to the same universality class.

V. SUMMARY AND CONCLUSIONS

In this paper we discussed the connection between a discrete growth model, the next-nearest-neighbor ballistic deposition, and a continuum equation, the Kardar-Parisi-Zhang equation. It has been believed for the last two decades [2] that the BD model belongs to the KPZ universality class; however, a formal derivation was lacking. In this work, we show that the absence of a formal derivation is not accidental, but rather reflects significant differences between the continuum equation that describes the BD model and the KPZ equation. This difference, which is mild in one dimension in the presence of noise, may become crucial in higher dimensions. It is also reflected in the different deterministic one-dimensional flattening of the surface, although this is not surprising. Our main conclusion is that mainly due to the form of our continuum equation in dimensions higher than 1, which breaks the rotation symmetry, the burden of proof has shifted and heavy simulations are needed, in order to obtain the BD exponents to an accuracy that will enable to answer whether it belongs to the KPZ universality class for $d>1$ or not. The goal of such simulations should be an accuracy compared to that in critical phenomena. If that is achieved, we will be able to tell with much higher certainty if the two models belong to the same universality class or not.

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